Variable separation approach for a differential-difference system: special Toda equation

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Abstract

A multi-linear variable separation approach is developed to solve a differential-difference Toda equation. The semi-discrete form of the continuous universal formula is found for a suitable potential of the differential-difference Toda system. Abundant semi-discrete localized coherent structures of the potential can be found by appropriately selecting the arbitrary functions of the semi-discrete form of the universal formula.

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1 Introduction

It is very difficult to find explicit exact solutions both for nonlinear partial differential equations (PDEs) and for nonlinear differential-difference equations (DDEs). In linear mathematical physics, the Fourier transform and the variable separation approach are two of the most effective ways to find exact solutions of linear equations. In nonlinear mathematical physics, the so-called inverse scattering transform (IST) has served as "nonlinear" Fourier transform for nonlinear integrable models. However, it is very hard to extend the variable separation approach to nonlinear cases even for integrable systems.

Recently, several kinds of "variable separation" approaches, say, the classical method, the differential Stäckel matrix approach [1], the geometric method [2], the ansatz-based method [3, 2], functional variable separation approach[4], the derivative dependent functional variable separation approach[5], the formal variable separation approach (nonlinearization of the Lax pairs or symmetry constraints) [6] and the multilinear variable separation approach (MVSA) [7]–[9] are in progress. Among these approaches, the MVSA may be the most powerful method to solve many (2+1)-dimensional systems both for integrable models[7]–[9] and for nonintegrable systems[10]. In [9], a quite universal formula

$$U \equiv \frac{2(a_1a_2 - a_0a_3)q_y p_x}{(a_0 + a_1p + a_2q + a_3pq)^2}, \tag{1}$$

is established to describe suitable physical quantities for various (2+1)-dimensional models solvable via MVSA. In (1), $a_0,\ a_1,\ a_2$ and a_3 are arbitrary constants and p is an arbitrary function of $\{x,\ t\}$ for all of the known MVSA solvable models, while q of (1) may be an arbitrary function of $\{y,\ t\}$ for some of MVSA solvable models, or an arbitrary solution of a Riccati equation for some other MVSA solvable models. Because some arbitrary characteristics, lower dimensional functions (like p), have been included in the universal formula (1), by selecting them appropriately, abundant localized structures like the multiple solitoffs, dromions, lumps, breathers, instantons, ring solitons and chaotic and fractal patterns have been found[9].

In this paper, we are interested in the following important question: Can the MVSA be extended to solve some nonlinear DDEs? In section 2, the MVSA is outlined for arbitrary DDEs. In sections 3 and 4, the MVSA is applied to a special differential-difference Toda equation (SDDTE). A semi-discrete form of the universal formula (1) for a suitable potential of the SDDTE is given in section 5. Starting from the semi-discrete form of the universal formula, the abundant semi-discrete localized excitations can be found for the potential of the SDDTE. Some special semi-discrete localized excitations of the model are also plotted in section 5. Section 6 contains a short summary and discussions.

2 Outline of the MVSA

Consider a given DDE of the following type

$$\begin{aligned}&F(x_i, n_j, u(x_i, n_j - k, ...), u_{x_i}(x_i, n_j - k, ...), ..., i = 1,\ ..., N_1, j = 1, ..., N_2, k = 0, \pm 1,\ \pm 2, ...)\\ &\equiv F(u) = 0,\end{aligned} \tag{2}$$

where $x_i,\ i = 1, 2, ...N_1$ are continuous variables and $n_j,\ j = 1, 2, ...N_2$ are discrete variables. As in the continuous case, the MVSA can be performed for DDEs via the following standard procedures.

(i). Multilinearize the original DDE (2) by using a suitable Bäcklund transformation. Usually, the resulting equations are the bilinear equations for integrable systems.

(ii). Choose a seed solution of the Bäcklund transformation as general as possible with one or more arbitrary functions.

(iii). Make a suitable variable separation ansatz with some variable separated functions. Usually, the ansatz is just the generalization of the Hirota's two-soliton solution.

(iv). Substitute the ansatz into the multilinear equations and separate the resulting equations to several variable separated ones.

(v). Solve the variable separated equations. Usually, to solve the variable separated PDEs and/or DDEs is still very difficult for any fixed seed solution. However, one can treat the problem in an alternative easy way: The variable separated functions appeared in the ansatz can be considered as arbitrary functions and then the function(s) appeared in the seed solution should be fixed from the variable separated equations.

(vi). Find a suitable field or potential which possesses a special variable separated solution described by the universal formula (1) for PDEs ($N_2 = 0$ in (2)) or a semi-discrete form of (1) for DDEs.

(vii). Discuss the possible semi-discrete localized excitations by selecting the arbitrary functions appropriately.

To see the details on the procedures of the MVSA for the DDE systems, we take the SDDTE as a simple concrete example in the remained sections.

3 SDDTE and its generalized bilinear form

In nonlinear discrete and semi-discrete physics systems, the most famous and important systems are the so-called Toda systems which are widely used in physics[11]. In this paper, we only consider a special differential-difference Toda equation (SDDTE)

$$Q(n)_{yt} = \exp[Q(n+1) - Q(n)][Q(n+1) + Q(n)]_y - \exp[Q(n) - Q(n-1)][Q(n) + Q(n-1)]_y, \quad (3)$$

where $Q(n) \equiv Q(n,\ y,\ t)$ is a function of the discrete variable n and the continuous variables $\{y,\ t\}$. The SDDTE (3) was firstly derived by Cao, Geng and Wu in a remarkable paper [12]. Some interesting integrable properties of the SDDTE (3) have been given in [12] and [13].

To solve the SDDTE (3) via MVSA, one can use the following dependent variable transformation

$$Q(n) = \rho(n) + \ln\left(\frac{f(n+1)}{f(n)}\right) \quad (4)$$

to bilinearize it. In the transformation (4), $\rho(n) \equiv \rho(n,t)$, the seed solution of the SDDTE has been selected as an arbitrary function of $\{n,\ t\}$ for convenience later.

Substitution of the dependent variable transformation (4) into the SDDTE yields

$$(T_+ - 1)\left(\frac{D_y D_t f(n)\cdot f(n) - 2\exp[\rho(n)-\rho(n-1)]D_y \exp(D_n) f(n)\cdot f(n)}{2f(n)^2}\right) = 0, \tag{5}$$

where T_+ is a shift operator, i.e.,$T_+F(n) = F(n+1)$, and the Hirota's bilinear differential operator $D_y^m D_t^k$ and the bilinear difference operator $\exp D_n$ are defined by

$$D_y^m D_t^k a\cdot b \equiv \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(y,t)b(y',t')\bigg|_{y=y',t=t'},$$

$$\exp(\delta D_n)a(n)\cdot b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n)\bigg|_{n=n'} \equiv a(n+\delta)b(n-\delta).$$

Multiplying (5) by the inverse operator of $T_+ - 1$ leads to a generalized bilinear SDDTE

$$D_y D_t f(n)\cdot f(n) - J(n,t)D_y \exp(D_n) f(n)\cdot f(n) + R(y,t)f(n)^2 = 0, \tag{6}$$

where

$$J(n) \equiv 2\exp[\rho(n)-\rho(n-1)] \tag{7}$$

is an arbitrary function of $\{n,\ t\}$ and $R(y,t)$, the kernel of the difference operator $T_+ - 1$, is an arbitrary function of $\{y,\ t\}$.

4 Variable separation solution of the SDDTE

In order to find some exact solution of (6), similar to the continuous cases[7]–[9], we look for the solutions of (6) in the form

$$f(n) = a_0 + a_1 p(n,t) + a_2 q(y,t) + a_3 p(n,t) q(y,t), \tag{8}$$

where $a_0,\ a_1,\ a_2$ and a_3 are arbitrary constants and the variable separation functions $q(y,t) \equiv q$ and $p(n,t) \equiv p(n)$ are only functions of $\{y,\ t\}$ and $\{n,\ t\}$ respectively. (8) looks like the Hirota's two soliton form when q and $p(n)$ are exponential functions. Substituting the ansatz (8) into (6), we have

$$\begin{aligned}&-2(a_2 + a_3 p(n))^2 q_y q_t + 2(a_3 a_0 - a_2 a_1) q_y p_{nt} + 2(a_2 + a_3 p(n))(a_0 + a_1 p(n) + a_2 q + a_3 p(n)) q_{yt}\\ &-J(n)(p(n+1) - p(n-1))(a_3 a_0 - a_2 a_1) q_y + R(a_0 + a_2 q + a_1 p(n) + a_3 q p(n))^2 = 0\end{aligned} \tag{9}$$

Because $p(n)$ and $J(n)$ are only functions of $\{n,\ t\}$ and q and R are only functions of $\{y,\ t\}$, (9) can be separated into the following two equations,

$$q_t = (a_0 + a_2 q)^2 c_1 + (a_1 + a_3 q)^2 c_2 + (a_0 + a_2 q)(a_1 + a_3 q) c_3, \tag{10}$$

$$p_t(n) = (a_0 a_3 - a_1 a - 2)(c_2 - c_3 p(n) + c_1 p(n)^2) + \frac{1}{2} J(n)(p(n+1) - p(n-1)), \tag{11}$$

when the arbitrary function R is selected as

$$R = -2(c_1 a_2^2 + c_2 a_3^2 + c_3 a_2 a_3) q_y, \tag{12}$$

with $c_1 \equiv c_1(t),\ c_2 \equiv c_2(t)$ and $c_3 \equiv c_3(t)$ being arbitrary functions of t.

In principle, as long as the arbitrary functions $c_1,\ c_2,\ c_3$ and $\rho(n)$ (and then $J(n)$) are fixed, we can obtain the corresponding special solutions of the (10) and (11) and then the special solutions of the SDDTE (3). However, it is still very difficult to solve the nonlinear DDE (11) for fixed nonzero $J(n)$. Fortunately, as in the continuous cases discussed in [7]–[9], because of the arbitrariness of the function $J(n)$, we can treat the problem alternatively. We can consider the function $p(n)$ as an arbitrary function of the variables n and t and fix the function $J(n)$ from the equation (11). The result reads

$$J(n) = \frac{2}{p(n-1) - p(n+1)} [(a_0 a_3 - a_1 a - 2)(c_2 - c_3 p(n) + c_1 p(n)^2) - p_t(n)]. \tag{13}$$

It should be pointed out that the Riccati equation (10) is totally same as that of the asymmetric Nizhnik-Novikov-Veselov (ANNV) equation[9]. To find out some special solutions of (10), one may select the arbitrary functions appropriately. Here we list two special selections.

(1). If we write $c_1,\ c_2$ and c_3 as

$$c_1 = \frac{a_3^2 A_{2t}}{(a_1 a_2 - a_0 a_3)^2} - \frac{a_3(a_1 + a_3 A_2) A_{1t}}{(a_1 a_2 - a_0 a_3)^2 A_1} - \frac{(a_1 + a_3 A_2)^2 A_{3t}}{(a_1 a_2 - a_0 a_3)^2 A_1}, \tag{14}$$

$$c_2 = \frac{a_2^2 A_{2t}}{(a_1 a_2 - a_0 a_3)^2} - \frac{a_2(a_0 + a_2 A_2) A_{1t}}{(a_1 a_2 - a_0 a_3)^2 A_1} - \frac{(a_0 + a_2 A_2)^2 A_{3t}}{(a_1 a_2 - a_0 a_3)^2 A_1}, \tag{15}$$

$$\begin{aligned} c_3 = {} & \frac{(a_0 a_3 + a_1 a_2 + 2 a_2 a_3 A_2) A_{1t}}{(a_1 a_2 - a_0 a_3)^2 A_1} - \frac{2 a_2 a_3 A_{2t}}{(a_1 a_2 - a_0 a_3)^2} \\ & + 2 \frac{(a_0 + a_2 A_2)(a_1 + a_3 A_2) A_{3t}}{(a_1 a_2 - A)^2 A_1} \end{aligned} \tag{16}$$

with $A_1 \equiv A_1(t), A_2 \equiv A_2(t)$ and $A_3 \equiv A_3(t)$ being arbitrary functions of t, then the general solution of (10) with (14)-(16) reads

$$q = \frac{A_1}{A_3 + F_1(y)} + A_2. \tag{17}$$

where $F_1 \equiv F_1(y)$ is an arbitrary function of y.

(2). If we select c_1, c_2 and c_3 as

$$c_1 = \frac{a_3^2 b_{0t}}{(a_1a_2-a_0a_3)^2} - \frac{a_3(a_1+a_3b_0)b_{1t}}{(a_1a_2-a_0a_3)^2 b_1} - \frac{[(a_1+a_3b_0)^2 - b_1^2a_3^2]b_{2t}}{(a_1a_2-a_0a_3)^2 b_1}, \tag{18}$$

$$c_2 = \frac{a_2^2 b_{0t}}{(a_1a_2-a_0a_3)^2} - \frac{a_2(a_0+a_2b_0)b_{1t}}{(a_1a_2-a_0a_3)^2 b_1} - \frac{[(a_0+a_2b_0)^2 - a_2^2b_1^2]b_{2t}}{(a_1a_2-a_0a_3)^2 b_1}, \tag{19}$$

$$\begin{aligned} c_3 = {} & \frac{(a_0a_3+a_1a_2+2a_2a_3b_0)b_{1t}}{(a_1a_2-a_0a_3)^2 b_1} - \frac{2a_2a_3b_{0t}}{(a_1a_2-a_0a_3)^2} \\ & + 2\frac{[(a_0+a_2b_0)(a_1+a_3b_0) - a_2a_3b_1^2]b_{2t}}{(a_1a_2-A)^2 b_1} \end{aligned} \tag{20}$$

with $b_0 \equiv b_0(t)$, $b_1 \equiv b_1(t)$ and $b_2 \equiv b_2(t)$ being arbitrary functions of t, then the general solution of (10) with (18)-(20) reads

$$q = b_1 \tanh(b_2 + F_2(y)) + b_0 \tag{21}$$

with $F_2 \equiv F_2(y)$ being an arbitrary function of y.

5 Abundant coherent structures for a potential of the SDDTE

Substituting all the results of the last section into (4), one can get many kinds of exact solutions for the field Q of the SDDTE. In continuous cases, it has been pointed out that for every of the MVSA solvable systems listed in [9], there exits a quantity that can be described by the universal formula (1). Now an important question is:

Is there a suitable potential for the SDDTE that can be described by a suitable semi-discrete form of the universal formula (1)?

Fortunately, it is straightforward to prove that if we define a potential of the SDDTE as

$$u \equiv -2Q_y(n), \tag{22}$$

then

$$u = U(n) \equiv \frac{2q_y(a_2a_1 - a_3a_0)(p(n+1)-p(n))}{(a_0+a_2q+a_1p(n)+a_3qp(n))(a_0+a_2q+a_1p(n+1)+a_3qp(n+1))}. \tag{23}$$

It is clear that the function $U(n)$ defined in (23) is just one suitable semi-discrete form of the continuous universal quantity U given in (1). We say the function $U(n)$ is semi-discrete means that it is discrete in one direction and continuous in other direction.

Now starting from the semi-discrete form of the universal quantity, we can obtain abundant semi-discrete localized excitations for the SDDTE by selecting the arbitrary functions appropriately.

The detailed studies show us that the semi-discrete localized structures for the potential u expressed by the semi-discrete form of the universal quantity are very similar to the continuous ones which have been discussed in Ref. [9]. So in this paper we will not discuss all the possible localized excitations but only list some particular examples.

Example 1. Resonant semi-discrete dromion and solitoff solutions.

If we restrict the functions $p(n)$ and q of (1) as

$$p(n) = \sum_{i=1}^{N} \exp(k_i n + \omega_i t + x_{0i}) \ \equiv \ \sum_{i=1}^{N} \exp(\xi_i) \tag{24}$$

$$q = \sum_{i=1}^{M} \exp(K_i y + y_{0i}) \sum_{j=1}^{J} \exp(\Omega_j t + t_{0j}), \tag{25}$$

where $x_{0i},\ y_{0i},\ t_{0j},\ k_i,\ \omega_i,\ K_i$ and Ω_i are arbitrary constants and $M,\ N$ and J are arbitrary positive integers, then we have a single resonant semi-discrete dromion solution or semi-discrete multiple solitoff solutions. The selection (25) is related to the selections on the functions of $A_i,\ i = 1,\ 2,\ 3,\ F_1$ of (17) as

$$A_3 = A_2 = 0, \tag{26}$$

$$A_1 = \sum_{j=1}^{J} \exp(\Omega_j t + t_{0j}), \tag{27}$$

$$F_1 = \frac{1}{\sum_{i=1}^{M} \exp(K_i y + y_{0i})}, \tag{28}$$

and the $c_i,\ i = 1,\ 2,\ 3$ are given by (14)–(16) with (26) and (27).

In Fig.1, we plot four typical structures caused by the resonant effects of four straight-line semi-discrete soliton solutions.

Fig. 1a shows the structure of a first type of single resonant semi-discrete dromion solution shown by (22) with (24), (25),

$$M = N = 2, J = k_1 = K_1 = 1,\ k_2 = K_2 = \frac{1}{3},\ a_0 = 1,\ a_1 = a_2 = 10,\ a_3 = \frac{1}{2} \tag{29}$$

and

$$x_{01} \ = \ y_{01} \ = t_{01} \ = \ x_{02} \ = y_{02} \ = \ 0 \tag{30}$$

at $t = 0$.

Fig. 1b is a plot of a single resonant semi-discrete solitoff solution shown by (22) with (24), (25), (30) and

$$M = N = 2, J = k_1 = K_1 = 1,\ k_2 = K_2 = \frac{1}{3},\ a_0 = 1,\ a_1 = a_2 = 3,\ a_3 = 0,\ t = 0. \tag{31}$$

Fig. 1c shows the structure of a second type of single semi-discrete resonant dromion solution shown by (22) with (24), (25), (30) and

$$M = N = 2, J = k_1 = -K_1 = 1,\ -k_2 = K_2 = \frac{1}{3},\ a_0 = a_2 = 10,\ a_3 = 1, a_1 = \frac{1}{2},\ t = 0. \tag{32}$$

Fig. 1d is a plot of a four solitoff solution shown by (22) with (24), (25), (30) and

$$M = N = 2, J = k_1 = -K_1 = 1,\ -k_2 = K_2 = \frac{1}{3},\ a_0 = 1,\ a_1 = a_2 = 3,\ a_3 = 0, t = 0. \tag{33}$$

Example 2. Semi-discrete oscillating dromions and lumps.

If some periodic functions in space variables are included in the functions $p(n)$ and q of (22), we may obtain some types of semi-discrete multi-dromion and multi-lump solutions with oscillating tails. The oscillated lump solution plotted in Fig. 2 is related to

$$q = \frac{1}{1 + [y(\cos(y) + 5/4)]^2},\ p(n) = \frac{1}{1 + (n - ct)^2}, \tag{34}$$

$$a_0 = a_3 = 1,\ a_1 = a_2 = 5 \tag{35}$$

at $t = 0$.

Example 3. Multiple ring soliton solutions.

In high dimensions, in addition to the point-like localized coherent excitations, there may be some other types of physically significant localized excitation. Recently, we have found some different kinds of ring soliton solutions which are not identically equal to zero at some closed (2+1)-dimensional and (3+1)-dimensional curves and decay exponentially away from the curves[14, 9, 15].

In Fig. 3, a typical saddle type semi-discrete ring soliton solution is plotted for the potential u with the selections

$$q = \exp\left(-\frac{y^2}{80} + 5\right),\ p(n) = \exp\left(\frac{(n - ct)^2}{80}\right), \tag{36}$$

and

$$a_0 = a_3 = 0,\ a_1 = a_2 = 5, \tag{37}$$

at $t = 0$.

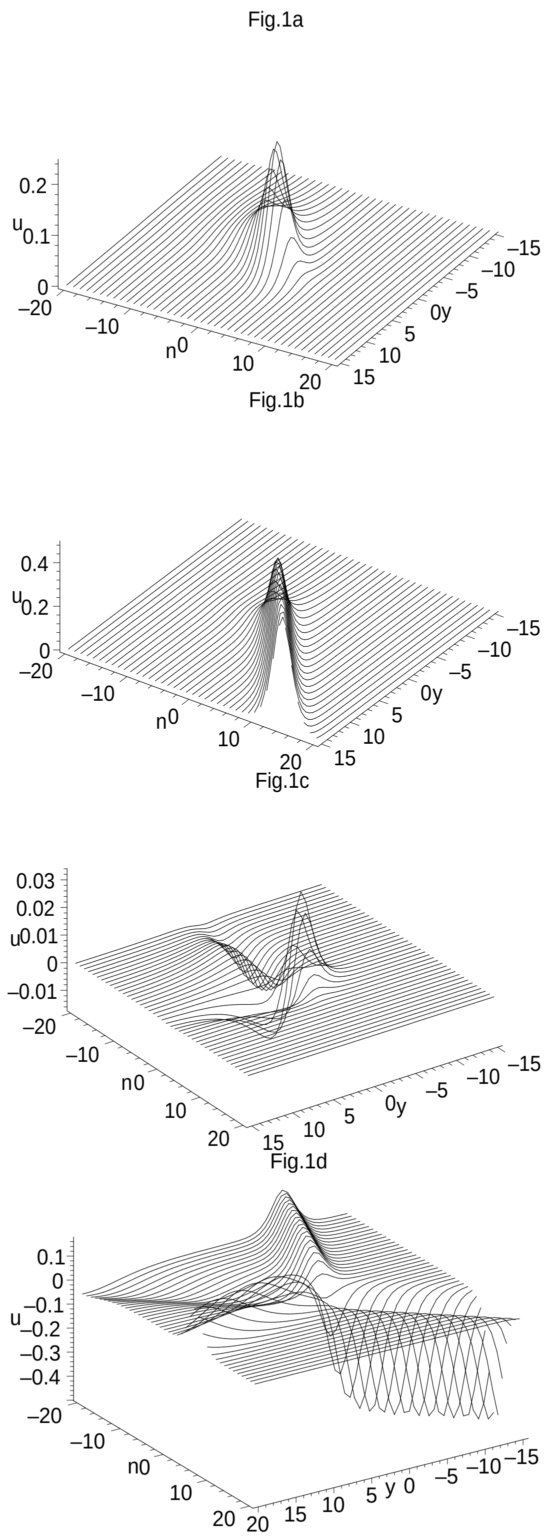

Figure 1: Four typical semi-discrete structures of SDDTE for the potential u expressed by (22) with (24) and (25). (a). A special single-peak semi-discrete resonant dromion solution. (b). A single semi-discrete solitoff solution. (d). A multi-peak semi-discrete dromion solution. (b). A semi-discrete four-solitoff solution.

Fig.2

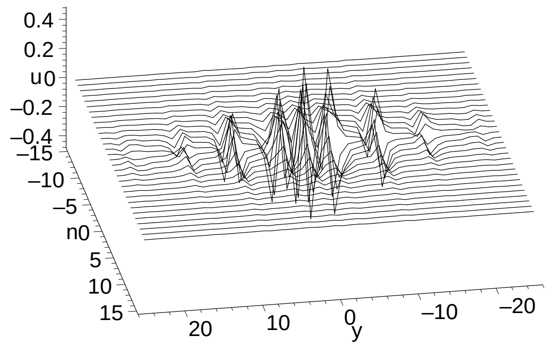

Figure 2: Plot of a special oscillating lump solution of the SDDTE for the potential u expressed by (22) with (34) and (35) at $t = 0$.

Fig.3

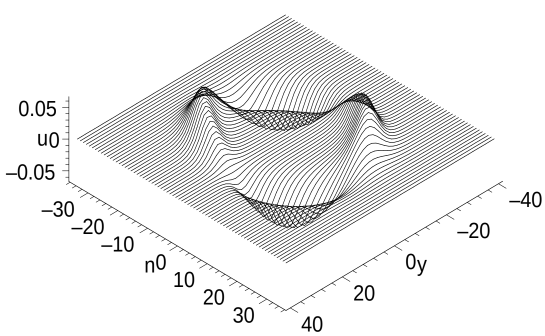

Figure 3: Plot of a typical single saddle type semi-discrete ring soliton solution for the potential u of the SDDTE with the selections (36) and (37) at $t = 0$.

Because of the existence of the arbitrariness in the expression (22) for the potential u, there exist many kinds of multiple semi-discrete ring soliton solutions. Since the situations are quite similar to those in the continuous cases, we do not discuss them further. However it is worth to mention that as pointed out in Refs. [9] and [10], the interactions among the travelling ring soliton solutions are completely elastic without phase shift.

6 Summary and discussion

In the previous studies[7]–[9], we have successfully solved several famous (2+1)-dimensional nonlinear continuous integrable models via a multilinear variable separation approach (MVSA). In this paper, we have further extended the MVSA to solve nonlinear differential-difference systems. Taking a special differential-difference Toda equation (SDDTA) as a concrete example, we have successfully solved the model via the MVSA.

In continuous cases, a quite universal formula has been found to describe suitable fields or potentials of MVSA solvable models. For the SDDTA, a semi-discrete form of the universal formula is found for a suitable potential. An arbitrary function related to the discrete space variable is included in the semi-discrete form of the universal formula. Another function included in the formula is an arbitrary solution of the Riccati equation and the Riccati equation is totally same as that in some continuous MVSA solvable models.

By selecting the arbitrary functions appropriately, one can find abundant semi-discrete localized excitations like the multiple solitoffs, dromions, lumps, breathers, instantons and ring soliton solutions. The semi-discrete localized solutions are quite similar to those of continuous cases shown in [9].

Though the MVSA have been successfully applied to the SDDTE, there are various interesting and important problems are worth to studying further. For instance, in continuous case, more than ten models have been solved by the MVSA (see Ref. [9] and the references therein). How many DDEs may be MVSA solvable? It is also known that for a continuous integrable model, there exist some different types of integrable discrete forms. Is the semi-discrete form of the universal formula unique? In other words, how universal the semi-discrete quantity (23) is? In addition to the DDEs, there may be various fully discrete systems in real physics. Can we extend the MVSA to solve some fully discrete nonlinear systems?

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